Week 5: Midterm Review!
MATH 4A

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Disclaimer: Since I am not the one writing the exam, I cannot guarantee this practice "exam" will look anything like the midterm. However, I reckon if you can do these without trouble, you're probably quite fine for the midterm.

Definitions: Here's a list of definitions that you should *definitely* know. Note that this list may not be comprehensive!

- A systems of equations is consistent when...?

  There is *a* solution. This means that some solution exists, although there may be more than one.

- A matrix is in RREF when...

  There is a pivot in every row, and the entries above the pivots are all 0.

- A matrix is invertible when...

  A matrix A is invertible if there's another matrix B such that $AB = BA = I$, where I is the identity matrix. Note that in this case, A must be a square.

- Consider the set of vectors $\{v_1, v_2, \cdots, v_n\}$ in $\mathbb{R}^m$. Define the span of these vectors.

  The set of linear combinations of $\{v_1, \cdots, v_n\}$. In other words, all such vectors of the form $c_1v_1 + \cdots + c_nv_n$, where $c_1, \cdots, c_n$ are all scalars.

- Consider the set of vectors $\{v_1, v_2, \cdots, v_n\}$ in $\mathbb{R}^m$. We say $\{v_1, v_2, \cdots, v_n\}$ is linearly independent when...

  If $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$, it's necessarily the case that $c_1 = c_2 = \cdots = c_n = 0$.

- What does it mean for $T : \mathbb{R}^n \to \mathbb{R}^m$ to be a linear transformation?

  For all vectors v_1, v_2, and scalars c, we have $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv_1) = cT(v_1)$.

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We say T is one-to-one when... Additionally, if T is one-to-one, what can we say about m in relation to n?

  For any v_1 and v_2 in $\mathbb{R}^n$, if $T(v_1) = T(v_2)$, we must have $v_1 = v_2$. In other words, different inputs get sent to different outputs. If T is one-to-one, we must have $n \leq m$.

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We say T is onto when... Additionally, if T is onto, what can we say about m in relation to n?

  For every y in $\mathbb{R}^m$ (the codomain), there's some x in $\mathbb{R}^n$ (the domain) where $T(x) = y$. In this case, $n \geq m$.

- What is a basis of a vectors space?

  A linearly independent set that spans the vector space.

- What is the dimension of a vector space?

  The dimension of a basis of the vector space.

- What is the nullity of a matrix? What is the null space (kernel)?

  Given a matrix A, the null space (kernel) of A consists of v such that $Av = 0$. This forms a vector space, the dimension of which is the nullity.

- What is the rank of a matrix? What is the column space (image)?

  Given a matrix A, the column space (image) of A consists of the outputs of A (ie. anything of the form Av, for v in the domain). Alternatively, this is the span of the columns of A. This again, forms a vector space, the dimension of which is the rank.

3-2.3 Find a set of vectors $\{u, v\}$ in $\mathbb{R}^4$ that spans the solution set of

$$\begin{cases} w - x + y - 2z &= 0, \\ 3w + 2x - y + z &= 0. \end{cases}$$

Solution: If try to solve for this systems of equation, the augmented matrix corresponding to this system is

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -2 & 0 \\ 3 & 2 & -1 & 1 & 0 \end{array}\right].$$

After row reducing into RREF, we get

$$\left[\begin{array}{cccc|c} 1 & 0 & 1/5 & -3/5 & 0 \\ 0 & 1 & -4/5 & 7/5 & 0 \end{array}\right].$$

This corresponds to the equations

$$w + \frac{1}{5}y - \frac{3}{5}z = 0$$

and

$$x - \frac{4}{5}y + \frac{7}{5}z = 0.$$

So, if we have a solution $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$, then from the w, x, y, z are constrained by the above equations. So, we can put w and x in terms of y and z, and get

$$\begin{bmatrix} -\frac{1}{5}y + \frac{3}{5}z \\ \frac{4}{5}y - \frac{7}{5}z \\ y \\ z \end{bmatrix}.$$

This can be written

$$\begin{bmatrix} -\frac{1}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} \frac{3}{5} \\ \frac{7}{5} \\ 0 \\ 1 \end{bmatrix} z,$$

in which case, we have $u = \begin{bmatrix} -\frac{1}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} \frac{3}{5} \\ \frac{7}{5} \\ 0 \\ 1 \end{bmatrix}$.

3-2.9 A $= \begin{bmatrix} -3 & 9 & -9 \\ -4 & 14 & -14 \\ 1 & -1 & 1 \end{bmatrix}$. Is it true that $Ax = b$ has a solution for every b?

Solution: Note that this is asking if A is onto.

If it's onto, then row reducing into RREF should yield a pivot in every row. However, $\begin{bmatrix} -3 & 9 & -9 \\ -4 & 14 & -14 \\ 1 & -1 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} -3 & 9 & -9 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. If we continue row reducing, it is clear that we will not get any nonzero entries in the third column, in which case, the third column does not have a pivot. So, A is not onto.

4-1.5 Let $v = \begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix}$, $u = \begin{bmatrix} -3 \\ -3 \\ 8+k \end{bmatrix}$, and $w = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$. The set $\{v, u, w\}$ is linearly independent unless $k =$?

Let $v = \begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix}$, $u = \begin{bmatrix} -3 \\ -3 \\ 8+k \end{bmatrix}$, and $w = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$. The set $\{v, u, w\}$ is linearly independent unless $k =$?

Solution:

$\{v, u, w\}$ is linearly independent if the following condition is met: $c_1 v + c_2 u + c_3 w = \vec{0}$ if and only if $c_1 = c_2 = c_3 = 0$.

Note that $\{v, w\}$ (ie. without u) is linearly independent, since v is not a multiple of w. So, in order to make this set linearly *dependent*, we must find $c_1 v + c_2 w = u$. In other words, the following system must be consistent:

$$c_1 \begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 8+k \end{bmatrix}$$

The augmented matrix corresponding to this system is

$$\left[\begin{array}{cc|c} -4 & -4 & -3 \\ -6 & -1 & -3 \\ -8 & 2 & 8+k \end{array}\right]$$

Reducing this into RREF, we get

$$\left[\begin{array}{cc|c} 1 & 0 & 3/4 \\ 0 & 1 & 3/10 \\ 0 & 0 & k+11 \end{array}\right].$$

The last equation corresponds to $k+11$, so $k = -11$ is what we need for this system to be consistent, in which case, $\{v, u, w\}$ linearly *dependent*. In other words, for $\{v, u, w\}$ to be linearly *independent*, we need $k \neq -11$.

4-2.5 Let $v_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Suppose $T(v_1) = \begin{bmatrix} -12 \\ 8 \end{bmatrix}$ and $T(v_2) = \begin{bmatrix} 19 \\ -9 \end{bmatrix}$. For an arbitrary vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$, find $T(v)$.

Solution: If we could find c_1 and c_2 such that $c_1v_1 + c_2v_2 = v$, then we would be done, since $T(v) = T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2) = c_1T(v_1) + c_2T(v_2)$.

So, let's find c_1 and c_2 such that

$$c_1 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We note that this is a systems of equations, with the corresponding augmented matrix $\left[\begin{array}{cc|c} -1 & 1 & x \\ -2 & 3 & y \end{array}\right]$.

Row reducing this to RREF yields $\begin{bmatrix} 1 & 0 & -3x+y \\ 0 & 1 & -2x+y \end{bmatrix}$. This tells us $c_1 = -3x + y$ and $c_2 = -2x + y$.

Thus, we see $T(v) = c_1T(v_1) + c_2T(v_2) = (-3x + y) \begin{bmatrix} -12 \\ 8 \end{bmatrix} + (-2x + y) \begin{bmatrix} 19 \\ -9 \end{bmatrix} = \begin{bmatrix} -2x + 7y \\ -6x - y \end{bmatrix}$.

5-2.12 Let $A = \begin{bmatrix} -1 & -3 & -2 \\ 1 & 3 & 2 \\ -2 & -6 & -4 \end{bmatrix}$. Find a basis for the null space (kernel) of A.

Solution: This is the set of $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $Av = 0$.

We note that if $Av = \vec{0}$, then we have

$$\begin{bmatrix} -1 & -3 & -2 \\ 1 & 3 & 2 \\ -2 & -6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x - 3y - 2z \\ x + 3y + 2z \\ -2x - 6y - 4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We note that what we have above is a systems of equations, and we are trying to solve for x, y, z. The augmented matrix corresponding to this system is

$$\left[\begin{array}{ccc|c} -1 & -3 & -2 & 0 \\ 1 & 3 & 2 & 0 \\ -2 & -6 & -4 & 0 \end{array}\right]$$

which row reduces to

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to $x + 3y + 2z = 0$, so if $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ was *any* solution, we must have $x = -3y - 2z$, so $v = \begin{bmatrix} -3y - 2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} z$. Since y and z were free variables, we see that they are "unconstrained" (ie. they can be any number). In other words, any solution would be of the form $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} z$, where y and z are scalars.

So, we see that $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis.